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ON AN AVERAGING METHOD IN DYNAMICS OF A RIGID BODY

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L.I. KONKINA

(Frunze)

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Motion of a gyrostat in a Newtonian gravity field is investigated near its resonance frequency using the canonical action-angle variables. An approximate solution is constructed by using the method of averaging of the Delaunay — Hill type [1].

We investigate the motion of a gyrostat about a fixed point in a Newtonian gravity field using the methods of the perturbation theory [2]. Rotation of a rigid body corresponding to the Euler — Poinsot case serves as the unperturbed motion.

Such a choice of the unperturbed motion is expedient either in the case of a rapidly spinning gyrostat, or when the body is sufficiently far from the center of attraction [3]. The methods of the perturbation theory are especially effective when the equations of perturbed motion are written in the Hamiltonian form using the canonical action-angle variables. The latter variables were studied in [4— 7] for the Euler— Poinsot case.

Let us first write the Hamiltonian of the problem in the action-angle variables [5]. The kinetic energy of the gyrostat is determined by the formula [8]

$$T = {}^1/_2\,(Ap^2 + Bq^2 + Cr^2) + J\,(P\alpha + q\beta + r\gamma) + {}^1/_2\,J^2\Omega^2$$

Here A, B and C are the principal moments of inertia of the rigid body: p, q and r are the angular velocity components relative to the moving axes; α, β and γ are the direction cosines of the rotor (gyro wheel) axis relative to the moving axes; J is the moment of inertia of the rotor relative to the axis of rotation; Ω is the relative angular velocity of the rotor $(\Omega = \text{const})$. On passing to the canonical action-angle variables where $I_1, I_2, I_3, \varphi_1, \varphi_2$ and φ_3 are equal to the correspon-ding variables L, G, I, h, v and f of [5], the kinetic energy of the gyrostat will assume the following form for the regions of rotational motion:

$$T = \frac{I_2^2}{2A}\left(1 - \frac{C-A}{C}\,\frac{\varkappa^2}{\varkappa^2+\lambda^2}\right) + \frac{J}{A}\left(\alpha S_1 + \frac{A}{B}\beta S_2 + \frac{A}{C}\gamma S_3\right)\Omega I_2 +$$

$$\frac{J^2}{A^2}\left(A\alpha^2 + \frac{A^2}{B}\beta^2 + \frac{A^2}{C}\gamma^2\right)\Omega^2$$

$$\varkappa^2 = \frac{C(B-A)}{A(C-B)}, \quad \lambda^2 = \frac{D^2(I_2 - I_3)}{I_2}, \quad D^2 = \frac{2\varkappa^2}{\sqrt{1+\varkappa^2}}$$

$$S_1 = P\sum_{n=0}^{\infty}\frac{q^{n+1/2}}{1+q^{2n+1}}\cos(2n+1)\varphi_3$$

$$S_2 = -P\sqrt{1+\varkappa^2}\sum_{n=0}^{\infty}\frac{q^{n+1/2}}{1-q^{2n+1}}\sin(2n+1)\varphi_3$$

$$S_3 = P\varkappa\left(\frac{1}{4} + \sum_{n=1}^{\infty} \frac{q^n}{1+q^{2n}} \cos 2n\varphi_3\right)$$

$$P = \frac{1}{\sqrt{\varkappa^2+\lambda^2}}, \quad q = \exp\left(-\pi \frac{\mathbf{K}'}{\mathbf{K}}\right)$$

where $\mathbf{K}$ and $\mathbf{K}'$ are complete elliptic integrals with moduli λ and $\lambda' = \sqrt{1-\lambda^2}$ respectively.

Let us now transform the force functions of the Newtonian gravity field which has the form

$$U = -\frac{3g}{2R}(A\gamma_1^2 + B\gamma_2^2 + C\gamma_3^2)$$

where γ_1, γ_2 and γ_3 are the direction cosines of the radius vector of the fixed point with the origin at the center of attraction.

Expressing γ_1, γ_2 and γ_3 in terms of the action-angle variables and using the Fourier expansions obtained in [5], we arrive at the following expressions:

$$\gamma_i = \frac{\sqrt{I_2^2 - I_1^2}}{I_2}(\sin\varphi_2 S_{1i} + \cos\varphi_2 S_{2i}) + \frac{I_1}{I_2} S_i \quad (i = 1, 2, 3)$$

$$S_{11} = -P\sum_{n=0}^{\infty} C_{n11}\sin(2n+1)\varphi_3, \quad S_{21} = P\sum_{n=0}^{\infty} C_{n21}\cos(2n+1)\varphi_3$$

$$S_{12} = -P\sqrt{1+\varkappa^2}\sum_{n=0}^{\infty} C_{n12}\cos(2n+1)\varphi_3$$

$$S_{22} = -P\sqrt{1+\varkappa^2}\sum_{n=0}^{\infty} C_{n22}\sin(2n+1)\varphi_3$$

$$S_{13} = -P\varkappa\sum_{n=1}^{\infty} C_{n13}\sin 2n\varphi_3, \quad S_{23} = P\varkappa\left[-\frac{1}{4\operatorname{sh}\sigma} + \sum_{n=1}^{\infty} C_{n23}\cos 2n\varphi_3\right]$$

$$C_{n11} = \frac{\rho\operatorname{ch}\sigma}{\nu}, \quad C_{n21} = \frac{\rho_1\operatorname{sh}\sigma}{\nu}, \quad C_{n12} = \frac{\rho_1\operatorname{ch}\sigma}{\nu_1}$$

$$C_{n22} = \frac{\rho\operatorname{sh}\sigma}{\nu_1}, \quad C_{n13} = \frac{\rho_2\operatorname{ch}\sigma}{\nu_2}, \quad C_{n23} = \frac{\rho_3\operatorname{sh}\sigma}{\nu_2}$$

$$\rho = q^{n+1/2}(1-q^{2n+1}), \quad \rho_1 = q^{n+1/2}(1+q^{2n+1}), \quad \rho_2 = q^n(1-q^{2n})$$

$$\rho_3' = q^n(1+q^{2n})$$

$$\nu = 1 - 2q^{2n+1}\operatorname{ch}2\sigma + q^{4n+2}, \quad \nu_1 = 1 + 2q^{2n+1}\operatorname{ch}2\sigma + q^{4n+2}$$

$$\nu_2 = 1 - 2q^{2n}\operatorname{ch}2\sigma + q^{4n}$$

$$\sigma = \frac{\pi}{2K}F\left(\operatorname{arctg}\frac{\varkappa}{\lambda}, \lambda'\right)$$

where $F(\varphi, \lambda)$ is an elliptic integral of the first kind.

Thus the perturbed Hamiltonian (the perturbations are caused by the presence of the rotor and by the Newtonian gravity field) can be written using the the action-angle variables in the form of a series suitable for use with the asymptotic methods.

Using the method of averaging, we can replace the system of equations of perturbed motion

$$\frac{\partial I_i}{\partial t} = -\frac{\partial H}{\partial \varphi_i}, \quad \frac{\partial \varphi_i}{\partial t} = \frac{\partial H}{\partial I_i} \quad (i = 1, 2, 3) \tag{1}$$

by a simplified system which allows integration in quadratures.

Introducing a small parameter ε, we can write the Hamiltonian function in the form

$$H = H_0 + \varepsilon H_1 + \ldots \tag{2}$$

where H_0 is the Hamiltonian function corresponding to the unperturbed motion and εH_1 is the perturbing function. The small parameter is introduced in the following manner:

$$\varepsilon = \frac{J}{A}, \quad k\varepsilon = \frac{3g}{2R}$$

where k is a finite quantity.

We explain the evolutionary properties of the perturbed motion of the gyrostat using one of the variants of the method of averaging over the rapid angle variable first introduced in celestial mechanics by Delaunay and Hill. This method is expedient in the case of "sharp commensurability" of the frequencies, i.e. in the neighborhood of the internal resonance of the system. Assuming that the unperturbed motion is near to the perturbed motion in terms of the angle variables φ_2 and φ_3, we introduce a new angle variable d which is called, in celestial mechanics, the Delaunay anomaly

$$d = k_1\varphi_2 + k_2\varphi_3 \tag{3}$$

where k_1 and k_2 are certain specified positive integrals. This variable characterises the "detuning" of the resonance.

Let us perform the canonical transformation of the variables I_i and φ_i to the new variables I_i^* and φ_i^* $(i = 1,2,3)$ using the relations

$$\begin{aligned} &I_1^* = I_1, \quad \varphi_1^* = \varphi_1 \\ &I_2^* = \frac{1}{k_2} I_2, \quad \varphi_2^* = d = k_1\varphi_2 + k_2\varphi_3 \\ &I_3^* = I_3 - \frac{k_2}{k_1} I_2, \quad \varphi_3^* = \varphi_3 \end{aligned} \tag{4}$$

From (3) we find

$$\varphi_2 = \frac{d}{k_1} - \frac{k_2}{k_1}\varphi_3 \tag{5}$$

Substituting (5) into the equation describing the perturbed function, we obtain

$$H(I_1, I_2, I_3, \varphi_1, \varphi_2, \varphi_3) = H_1(I_1, I_2, I_3, d, \varphi_3)$$

The averaged value of the Hamiltonian is obtained by computing its mean value with respect to φ_3 with the remaining variables kept fixed. Using (4) we write the averaged value of the perturbed function in the form

$$\overline{H}_1^* = H_0 + \varepsilon\left\{\frac{Ak_2I_2^*\gamma\varkappa}{4C} + k\left[\frac{k_2I_2^{*2} - I_1^2}{k_2I_2^{*2}} p^2\left[\sum_{n=0}^{\infty}\lambda_{n1} + \right.\right.\right. \tag{6}$$

$$+\left(\sum_{n=0}^{\infty}\lambda_{n3}+\sum_{n=1}^{\infty}\lambda_{n4}\right)\left[\sin 2\left(\frac{d}{k_1}-\frac{k_2}{k_1}2\pi\right)-\sin\frac{2d}{k_1}\right]\Bigg]+$$

$$\frac{2I_1\sqrt{k_2^2I_2^{*2}-I_1^2}}{k_2^2I_2^{*2}}\left[\sum_{n=0}^{\infty}\lambda_{n5}+\sum_{n=1}^{\infty}\lambda_{n6}\right]\left[\sin\left(\frac{d}{k_1}-\frac{k_2}{k_1}2\pi\right)-\right.$$

$$\left.\sin\frac{d}{k_1}\right]+\frac{I_1^2}{2k_2^2I_2^{*2}}\left[\sum_{n=0}^{\infty}\lambda_{n7}+\sum_{n=1}^{\infty}\lambda_{n8}\right]\Bigg]\Bigg\}$$

$$H_0=\frac{k_2^2I_2^{*2}}{2A}\left(k-\frac{C-A}{C}\;\frac{\varkappa^2}{\varkappa^2+\lambda^2}\right)$$

$$\lambda_{n1}=A\,(C^2_{n11}+C^2_{n21})+$$

$$B\,(1+\varkappa^2)\,(C^2_{n12}+C^2_{n22}),\quad \lambda_{n2}=C\varkappa^2\left[C^2_{n13}+C^2_{n23}+\frac{1}{8\,\mathrm{sh}^2\,\sigma}\right]$$

$$\lambda_{n3}=A\left[\frac{k_1\,(C^2_{n11}-C^2_{n21})}{4\pi}\;\frac{1-k_2k^*}{k_2}-\frac{C_{n11}\,C_{n21}k_1^*}{8\pi}\right]+$$

$$B\,(1+\varkappa^2)\left[\frac{k_1\,(C^2_{n12}-C^2_{n22})}{4\pi}\;\frac{1-k_2k_1^*}{k_2}-\frac{C_{n12}C_{n22}k_1^*}{8\pi}\right]$$

$$\lambda_{n4}=C\varkappa^2\left[\frac{k_1\,(C^2_{n13}-C^2_{n23})}{4\pi}\;\frac{1-k_2k_2^*}{k_2}-\right.$$

$$\left.\frac{k_3^*\,(k_1nC_{n13}+k_2C_{n23})}{4\pi\,\mathrm{sh}\,\sigma}-\frac{k_3^*k_1nC_{n13}C_{n23}}{\pi}\right]$$

$$\lambda_{n5}=A\left(\frac{k_1C_{n21}C_{n1}}{4\pi k_2}+\frac{C_{n11}C_{n1}k_1^*}{2\pi}\right)+B\,(1+\varkappa^2)\times$$

$$\left(\frac{C_{n12}C_{n2}k_1^*}{4\pi}-\frac{C_{n22}C_{n2}k^*k_1}{4\pi}-\frac{k_1C_{n22}C_{n2}}{4\pi k_2}\right)$$

$$\lambda_{n6}=\frac{k_3^*}{8\pi}\left(k_1nC_{n13}+\frac{k_2}{\mathrm{sh}\,\sigma}\,C_{n3}+k_2C_{n23}\right)-\frac{2nk_1^*C_{n13}C_{n3}}{\pi\,(k_2^2-16n^2k_1^2)}-$$

$$\frac{k_1}{4\pi k_2}\left(\frac{1}{\mathrm{sh}\,\sigma}+C_{n23}C_{n3}\right)$$

$$\lambda_{n7}=AC^2_{n1}+B\,(1+\varkappa^2)\,C_{n2},\quad \lambda_{n8}=C\varkappa^2\,(1/6+C^2_{n3})$$

$$k^*=\frac{k_2}{k_2^2-k_1^2\,(2n+1)^2},\quad k_1^*=\frac{k_1^2\,(2n+1)}{k_2^2-k_1^2\,(2n+1)^2}$$

$$k_2^*=\frac{k_2}{k_2^2-k_1^2 4n^2},\quad k_3^*=\frac{k_1}{k_2^2-k_1^2n^2}$$

The averaged equations of the perturbed motion have the following structures:

$$\frac{\partial I_i^*}{\partial t}=0,\quad \frac{\partial \overline{I_2^*}}{\partial t}=\varepsilon\Phi_1\,(I_1^*,\ I_2^*,\ I_3^*,\ d)\quad (i=1,\ 3) \tag{7}$$

$$\frac{\partial\bar{\varphi}_1^*}{\partial t} = \varepsilon\Phi_2(I_1^*,\ I_2^*,\ I_3^*,\ d),\quad \frac{\partial\bar{\varphi}_i^*}{\partial t} = \omega_i + \varepsilon\Phi_3(I_1^*,\ I_2^*,\ I_3^*,\ d)\ (i=2,3)$$

$$\omega_2 = \frac{k_2 I_2^*}{A} - \frac{C-A}{2AC}\frac{I_2^*\varkappa^2}{\eta}[(D^2(2k_1k_2-3) - 2k_1k_2\varkappa^2)I_2^* - 3D^2I_3^*k_1]$$

$$\omega_3 = \frac{A-C}{2AC}\frac{k_1I_3^*\varkappa^2D^2}{\eta},\quad \eta = [I_2^*(D^2(1-k_1k_2)+\varkappa^2) + D^2k_1I_3^*]^2$$

The averaged system of equations (7) has three first integrals

$$I_1^* = C_1,\quad I_3^* = C_2,\quad \overline{H}_1^*(I_1^*,\ I_2^*,\ I_3^*,\ \varepsilon,\ d) = C_3 \tag{8}$$

and using (8) we reduce the problem to quadratures. The third (transcendental) relation of (8) yields the action in the form of a series in ε

$$I_2^* = I_2^*(d,\ \varepsilon) \tag{9}$$

The relation (9) can be written as a Fourier series the coefficients of which are power series in ε. Clearly, when the proposed method of solving the problem is used, the expression for I_2^* will not contain secular perturbations but only the long-period perturbations through the anomaly d. The coefficients of the long-period perturbations will be the greater, the "sharper" the commensurability of the frequencies.

Thus the differential equation for $\bar{\varphi}_2^*$, yields the relation

$$t = t_0 = \int_{\bar{\varphi}_{20}^*}^{\bar{\varphi}_2^*} \frac{d\varphi_2^*}{\omega_2 + \overline{\Phi}_3(I_1^*,\ I_2^*,\ I_3^*,\ \varphi_2^*)}$$

Now $\bar{\varphi}_2^*$ and I_2^* are known functions of time. This enables us to obtain $\bar{\varphi}_1^*$ and $\bar{\varphi}_3^*$ as functions of time

$$\bar{\varphi}_i^* - \bar{\varphi}_{i0}^* = \int_0^t \Phi_i(t)\,dt \quad (i=1,3)$$

In accordance with the proofs of the asymptotic methods [1], the inequalities $|I_i - \dot{I}_i| < \mu$ hold for any $\mu > 0$, for some $\varepsilon_0(\mu)$ and for all $\varepsilon \in [0,\ \varepsilon_0]$ and $t \in [0,\ 1/\sqrt{\varepsilon}]$ The canonical variables I_i characterize the behavior of the perturbed rotors. At relatively small ε the trajectories of perturbed motion wind themselves onto slightly deformed toruses.

Thus the solutions of the averaged equations obtained represent qualitatively, as well as quantitatively, with a great degree of accuracy, the actual motion of a gyrostat in a Newtonian gravity field in the region of a single internal resonance.

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REFERENCES

1. Grebenikov, E.A. and Riabov, Iu.A., New Qualitative Methods in Celestial Mechanics. Moscow, "Nauka", 1971.

2. Duboshin, G.N., Celestial Mechanics. Basic Problems and Methods. Moscow, "Nauka", 1968.

3. Demin, V.G. and Kiselev, F.I., A new class of periodic motions of a rigid body with a fixed point in a Newtonian force field. Dokl. Akad. Nauk SSSR, Vol. 214, No. 5, 1974.

4. Born, M., Problems of Atomic Dynamics. Vol. 1, N.Y. Ungar, 1960.

5. Sadov, Iu.A., The action - angle variables in the Euler — Poinsot Problem. PMM, Vol. 34, No. 5, 1970.

6. Boigey, F., Une thèorie des perturbations en variables angles — actions. Application au mouvement d'un solide autour d'un point fixe. Précession — nutation. J. méc., Vol. 11, No. 3, 1972.

7. Kinoshita, H., Stationary motions of triaxial body and their stabilities. Publ. Astron. Soc. Japan, Vol. 24, No. 3, 1972.

8. Rumiantsev, V.V., On the stability of the steady motions of a satellite. Moscow, Izd. VTs Akad. Nauk SSSR, 1967.

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